CSIR NET/JRF

Mathematical Science

30 Nov. 2020

PART-A

(Mathematical Sciences)

(1.) A $3.1\text{m} \times 2.2\text{m} \times 2.1\text{m}$ block of granite is cut to have maximum number of $3\text{m} \times 2\text{m}$ sized slabs having thickness of 4 cm. These slabs are used to make a 1.5m wide pavement. What is the maximum length (in meters) of pavement that can be made using these slabs?

(a.) 200
(b.) 220
(c.) 400
(d.) 440

(2.) Milk rises rapidly at boiling point because

(a.) It becomes hotter than the vessel at boiling point
(b.) Its temperature rapidly increases at boiling point
(c.) Its bulk density rapidly decreases and it becomes more buoyant at boiling point
(d.) Its vapour pressure rapidly decreases at boiling point

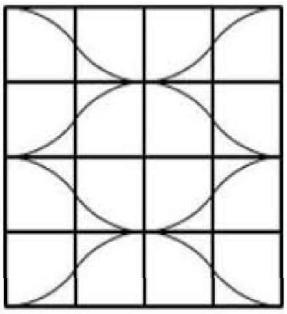
(3.) In an examination, each of the two brilliant student, got 100 out of 100 and each of the remaining six student, scored less than 12. There is no provision of getting negative marks. If $N =$ Median score of the students and $M =$ Mean score of the students which of the following is true?

(a.) $M > 2N$
(b.) $3N/2 < M \le 2N$
(c.) $N \le M \le 3N/2$
(d.) From the above information can be said about the relationship between the Median score of the students and the Mean score of the students

(4.) Along the common radial direction from the center of two concentric circles of radii 100 m and 150 m, point A is on the circumference of the inner circle and point B is on the circumference of the outer circle. The points A and B start moving on the respective circles with a speed of 8 m/s, at the same instant, but in opposite directions. After how many seconds, approximately, would they again cross each other along a common radial direction?

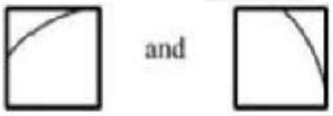
(a.) 31
(b.) 37
(c.) 47
(d.) 53

(5.)



Which combination of square tiles shown below can produce the given pattern on the floor?

(a.) 

(b.) 

(c.) 

(d.) 

(6.) Fifteen females participate in a singles badminton tournament. If a player is eliminated as soon as she losses a match, how many matches are required to determine the winner?

(a.) 30

(b.) 29

(c.) 15

(d.) 14

(7.) A box contains 3 white and 5 black balls. What is the probability of choosing 1 white and 1 black ball, if two balls are drawn at random, one by one, with replacement?

(a.) 15/64

(b.) 15/32

(c.) 3/32

(d.) 3/28

(8.) The first day of the year 2020 was a Wednesday. The first day of 2021 would be a

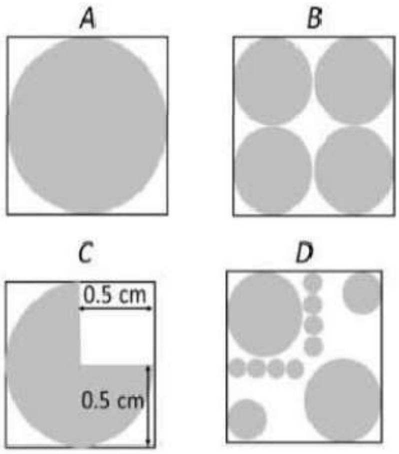
(a.) Wednesday

(b.) Thursday

(c.) Friday

(d.) Monday

(9.) The shaded circles having diameter of 1, 0.5, 0.25 and 0.125 cm are inside squares of side 1 cm. The ratio of shaded area in A and B is one. The ratio of shaded area in C and D would be



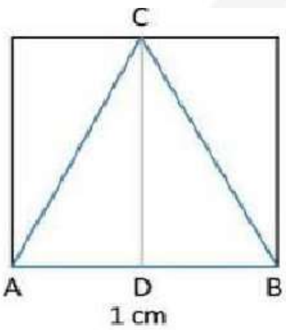
(a.) 0.75

(b.) 1

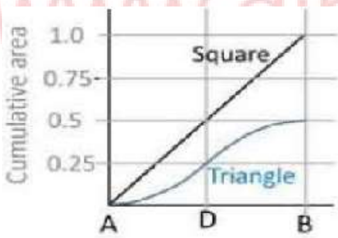
(c.) 1.25

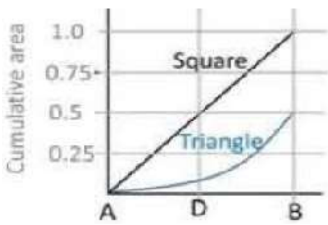
(d.) 1.5

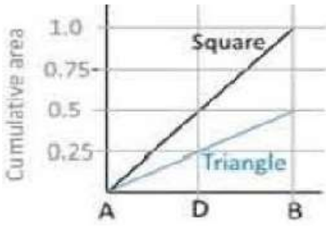
(10.) An isosceles triangle, ABC, is inside a square of side of 1cm where $AD = BD$.

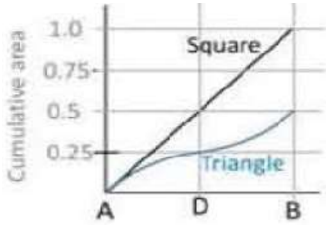


Which one of the following graphs represents cumulative area (cm[2]) of the square and the triangle when moving from A to B?

(a.)


(b.)


(c.) 

(d.) 

(11.) The coding genes form a tiny fraction of about 2% of the total human genome. If the total human genome is written on a pack of 50 cards, the number of cards corresponding to coding genes would be

(a.) One card

(b.) Two cards

(c.) Three cards

(d.) Four cards

(12.) Four locations A, B, C and D are along a straight road. Distance between A and C is 30 km. From A, a bus starts at 8 am to reach B, and another bus from C starts at 8:30 am to reach D. The buses move away from A, in the same direction with speed of the first being double of the second. They meet at 9 am, at a place covering 80% of the respective distances they are to travel. Then what is the distance between B and D?

(a.) 37.5 km

(b.) 25.0 km

(c.) 12.5 km

(d.) 7.5 km

(13.) Probability of selection of Alex for a post is 8/11 and for Gafoor it is 5/14. The selection of one is independent of the other. What is the probability of selection of only one of them for the post?

(a.) 25/72

(b.) 67/154

(c.) 87/154

(d.) 15/72

(14.) Suppose the standard deviation of the body temperatures of 17 persons measured in degrees Fahrenheit is 2.7. Which of the following is the standard deviation of the above 17 temperatures measured in degrees Celsius?

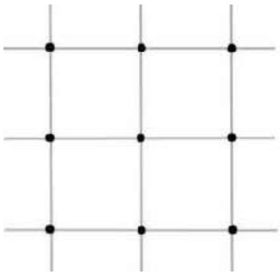
(a.) 3.5

(b.) 2.7

(c.) 2.5

(d.) 1.5

(15.) Dots are placed at the intersections of a grid.



How many triangles one can draw by joining these dots?

(a.) 74

(b.) 76

(c.) 78

(d.) 84

(16.) Three liquids, A, B and C having densities of 3, 2 and 1 g/cc respectively, are mixed in volume proportion of 1:2:3 to form a 6 ml solution. What will be the density (in g/cc) of the solution?

(a.) 1.7

(b.) 2.0

(c.) 2.5

(d.) 3.0

(17.) A fair coin is tossed 8 times independently. What is the probability that the third toss results in a head?

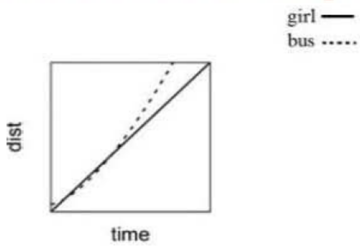
(a.) 1/8

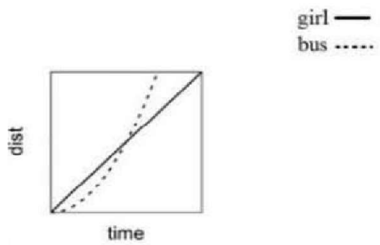
(b.) 1/4

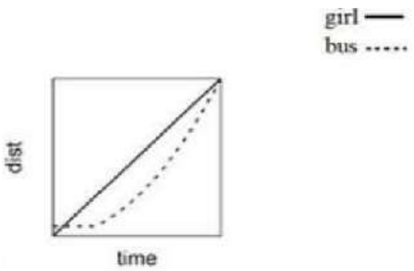
(c.) 1/2

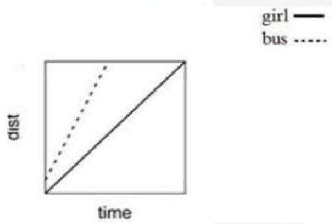
(d.) $3/2^8$

(18.) A girl is running at a constant speed along a straight path to catch a bus that is some distance away from her and stationary. Before she reaches the bus, it accelerates away from her. Which of the following graphs is a possible depiction of their motion?

(a.) 

(b.) 

(c.) 

(d.) 

(19.) A tap takes 6 hours to fill a water tank, a second tap takes 5 hours, a third 4 hours and a fourth 3 hours. How long, approximately, would it take to fill the tank if all the taps are used together?

(a.) 1 hour

(b.) 1.5 hours

(c.) 2 hours

(d.) 2.5 hours

(20.) The number of unit squares that can be fitted inside a circle of some finite diameter d is at the most

(a.) $(\pi d^2/4)-1$

(b.) πd^2

(c.) $(\pi d^2)/4$

(d.) $(\pi(d-1)^2)/4$

PART-B
(Mathematical Sciences)

(21.) Suppose that A, B are two non-empty subsets of $\mathbb{R}$ and $C = A \cap B$. Which of the following conditions imply that C is empty?

(a.) A and B are open and C is compact

(b.) A and B are open and C is closed

(c.) A and B are both dense in $\mathbb{R}$

(d.) A is open and B is compact

(22.) $\lim\limits_{n\to\infty} \dfrac{((n+1)(n+2)\ldots(n+n))^{1/n}}{n}$

(a.) Is equal to $\dfrac{e}{4}$

(b.) Is equal to $\dfrac{4}{e}$

(c.) Is equal to e

(d.) Does not exist

(23.) Let $Y = \{1,2,3,\ldots,100\}$ and let $h: Y \to Y$ be a strictly increasing function. The total number of functions $g: Y \to Y$ satisfying $g(h(j)) = h(g(j)), \forall j \in Y$ is

(a.) 0

(b.) 100!

(c.) 100^{100}

(d.) 100^{98}

(24.) An infinite binary word a is a string $(a_1 a_2 a_3 \ldots)$ where each $a_n \in \{0,1\}$. Fix a word $s = (s_1 s_2 s_3 \ldots)$, where $s_n = 1$ if and only if n is prime. Let $S = \{a = (a_1 a_2 s_3 \ldots) \mid \exists m \in \mathbb{N}$ such that $a_n = s_n, \forall n \ge m\}$. What is the cardinality of S?

(a.) 1

(b.) Finite but more than 1

(c.) Countably infinite

(d.) Uncountable

(25.) Let f be a non-constant polynomial of degree k and let $g: \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Which of the following statements is necessarily true?

(a.) There always exists $x_0 \in \mathbb{R}$ such that $f(x_0) = g(x_0)$

(b.) There is no $x_0 \in \mathbb{R}$ such that $f(x_0) = g(x_0)$

(c.) There is $x_0 \in \mathbb{R}$ such that $f(x_0) = g(x_0)$ if k is even (d.) There is $x_0 \in \mathbb{R}$ such that $f(x_0) = g(x_0)$ if k is odd

(26.) The sum of the finite series

$S = \frac{1}{2} - \frac{1}{3\times 1!} + \frac{1}{4\times 2!} - \frac{1}{5\times 3!} + \ldots$ is equal to

(a.) $2 - \frac{1}{e}$

(b.) $1 - \frac{2}{e}$

(c.) $\frac{2}{e} - 1$

(d.) $\frac{1}{e} - 2$

(27.) Let $S = \{u_1, \ldots, u_k\}$ be a subset of non-zero vectors from $\mathbb{R}^n$. Now, consider the two statements given below:

I. If S is linearly dependent in $\mathbb{R}^n$ then u_k is a linear combination of $u_1, \ldots, u_{k-1}$.

II. If S is linearly independent set in $\mathbb{R}^n$ then $k < n$.

Which of the following statements is true?

(a.) Statement I is FALSE and Statement II is TRUE

(b.) Statement I is TRUE end Statement II is FALSE

(c.) Both Statement I and Statement II are FALSE

(d.) Both Statement I and Statement II are TRUE

(28.) Let M be a 5×5 matrix with real entries such that $\text{Rank}(M) = 3$. Consider the linear system $Mx = b$. Let the row-reduced echelon form of the augmented matrix $[M\ b]$ be R and let $R[i,:]$ denote the i^{th} row of R. Suppose that the linear system admits a solution. Which of the following statements is necessarily true?

(a.) $R[3,\ :] = [0\ \ 1\ \ 0\ \ *\ \ *\ \ *]$

(b.) $R[5,\ :] = [0\ \ 0\ \ 1\ \ 0\ \ *\ \ *]$

(c.) $R[4,\ :] = [0\ \ 0\ \ 0\ \ 1\ \ *\ \ *]$

(d.) $R[4,\ :] = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]$

(29.) Let A be an $n\times n$ matrix of rank 1. Let $\alpha = \det(I + A)$, where I is the identity matrix and let $\beta = \text{trace } A$. Which of the following is true?

(a.) $\beta - \alpha = 1$

(b.) $\alpha - \beta = 1$

(c.) $\alpha < \beta + 1$

(d.) $\alpha > \beta + 1$

(30.) Let a, b and c be distinct integers. Let A be the matrix $A = \begin{pmatrix} a^2 & b^2 & c^2 \\ a^5 & b^5 & c^5 \\ a^{11} & b^{11} & c^{11} \end{pmatrix}$. Which among the following is the set of all possible ranks of A?

(a.) $\{3\}$

(b.) $\{2,3\}$

(c.) $\{1,2,3\}$

(d.) $\{0,1,2,3\}$

(31.) Which of the following is an inner product on the vector space of all real valued continuous functions on $[0,1]$?

(a.) $\langle f, g\rangle = \left|\int_0^1 f(t)g(t)\,dt\right|$

(b.) $\langle f, g\rangle = \int_0^1 |f(t)g(t)|\,dt$

(c.) $\langle f, g\rangle = f(0)g(0) + f(1)g(1)$

(d.) $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$

(32.) Let us define a matrix $A \in M_n(\mathbb{R})$ to be positive if for every column vector $v \in \mathbb{R}^n$ we have $\langle Av, v\rangle \geq 0$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. Let $A_{\alpha,\beta} = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & \beta \end{pmatrix}$. Let $S = \{(\alpha, \beta) \in \mathbb{R}^2 : A_{\alpha,\beta} \text{ is positive}\}$. Which of the following statements is true?

(a.) S is empty

(b.) $(\alpha,\beta) \in S$ if and only if $\alpha\beta > 0$

(c.) $(\alpha,\beta) \in S$ if and only if $\alpha + \beta + 4 > 0$

(d.) $S = \mathbb{R}^2$

(33.) A function $f : \mathbb{C} \mapsto \mathbb{C}$ is said to be analytic at ∞, if the function g defined by $g(w) = f\left(\frac{1}{w}\right)$ is analytic at 0 with an appropriate value given for $g(0)$. Which of the following statements is true?

(a.) Any non-constant polynomial is analytic at ∞

(b.) If f is analytic at ∞ then f is bounded

(c.) For any z_0 in $\mathbb{C}$, the function $f(z) = e^{1/z - z_0}$ is analytic at ∞

(d.) Any entire function can be extended to an analytic function at ∞

(34.) Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \left||z|^2 - 1\right|^2$.

Which of the following statements is true?

(a.) f is complex differentiable at all complex number except for $z \in \{0,1\}$

(b.) f is complex differentiable only at $z = 0$

(c.) f is complex differentiable only at $z = 1$

(d.) f is complex differentiable only for $z \in \{0,1\}$

(35.) Let f, g be entire functions such that $\lim_{z\to\infty} \frac{f(z)}{z^n} = \lim_{z\to\infty} \frac{g(z)}{z^n} = 1$ for some fixed positive integer n. Which of the following statements is true?

(a.) $f = g$

(b.) $f - g$ is necessarily a polynomial of degree at most $n-1$

(c.) There exist f, g with these properties such that $f - g$ is a polynomial of degree n

(d.) There exist f, g with these properties such that $f - g$ is not a polynomial

(36.) Let f be a holomorphic function on the disc $\{z \in \mathbb{C} : |z| < 2\}$. Assume that the only zero of f in the closed unit disc $\{z \in \mathbb{C} : |z| \le 1\}$ is a simple zero at the origin. Let γ be the positively oriented circle $\{z \in \mathbb{C} : |z| = 1\}$. The integral $\int_\gamma \frac{dz}{f(z)}$ equals

(a.) $2\pi i f'(0)$

(b.) $2\pi i f''(0)$

(c.) $2\pi i / f'(0)$

(d.) $2\pi i / f''(0)$

(37.) The last two digits of 3^{2019} are

(a.) 27

(b.) 37

(c.) 57

(d.) 67

(38.) Let G be a group of order 2020. Which of the following statements is necessarily true?

(a.) G is not a simple group

(b.) G has exactly four proper subgroups

(c.) G is a cyclic group

(d.) G is abelian

(39.) Which of the following statements is NOT true?

(a.) The polynomial ring $\mathbb{Z}[x]$ is a Principal Ideal Domain (PID)

(b.) The polynomial ring $\mathbb{Q}[x]$ is a Principal Ideal Domain (PID)

(c.) The polynomial ring $\mathbb{Z}[x]$ is a Unique Factorization Domain (UFD)

(d.) The polynomial ring $\mathbb{Q}[x]$ is a Unique Factorization Domain (UFD)

(40.) Suppose $f:[0,1]\times[0,1]\to(0,1)\times(0,1)$ is continuous non-constant function. Which of the following statements is NOT true?

(a.) Image of f is uncountable

(b.) Image of f is a path connected set

(c.) Image of f is a compact set

(d.) Image of f has non-empty interior

(41.) Consider the ODE $t\dot{y}-3y=t^2y^{1/2}$, $y(1)=1$. Find the value of $y(2)$

(a.) 14
(b.) 16
(c.) 0
(d.) 8

(42.) For $\lambda\in\mathbb{R}$, consider the system of differential equations

$x_1'=x_1+2x_2+2x_3$,

$x_2'=2x_1+x_3$,

$x_3'=-x_3+2x_2+\lambda x_3$.

If $\vec{x}(t)=\vec{a}\,te^{2t}$ (for some $\vec{a}$) is a solution of the system then the value of λ is equal to

(a.) 2
(b.) 4
(c.) 6
(d.) 1

(43.) Which of the following is solution to $u_x+x^2u_y=0$ with $u(x,0)=e^x$?

(a.) e^x

(b.) $e^{(x^3+y)^{1/3}}$

(c.) $e^{(x^3-3y)^{1/3}}$

(d.) $(x^2y+1)e^x$

(44.) Consider the differential equation

$x^2y''-2x(x+1)y'+2(x+1)y=0$.

If a polynomial is a solution then the degree of the polynomial is equal to

(a.) 1

(b.) 2
(c.) 3
(d.) 4

(45.) Consider the Newton-Raphson method applied to approximate the square root of a positive number α . A recurs ion relation for the error $e_n = x_n - \sqrt{\alpha}$ is given by

(a.) $e_{n+1} = \frac{1}{2}\left(e_n + \frac{\alpha}{e_n}\right)$

(b.) $e_{n+1} = \frac{1}{2}\left(e_n - \frac{\alpha}{e_n}\right)$

(c.) $e_{n+1} = \frac{1}{2}\frac{e_n^2}{e_n + \sqrt{a}}$

(d.) $e_{n+1} = \frac{e_n^2}{e_n + 2\sqrt{a}}$

(46.) The Euler equations satisfied by the extremals of the functional $J(y) = \int_0^5 \left[y^2 + x^2 y'\right] dx$ define a solution curve in the (x, y)-plane which is

(a.) Linear
(b.) Quadratic
(c.) Cubic
(d.) Trigonometric

(47.) Consider continuous solutions f of the following integral equation is $[0,1]$.

$f^2(t) = 1 + 2\int_0^t f(s)\,ds, \forall t \in [0,1]$.

Which of the following statements is true?

(a.) There is no solution
(b.) There is exactly one solution
(c.) There are exactly two solutions
(d.) There are more than two solutions

(48.) A body moves freely in a uniform gravitational field. The trajectory lies on which of the following curves in phase space?

(a.) Straight line in phase space
(b.) Parabola in phase space
(c.) Hyperbola in phase space
(d.) Ellipse in phase space

(49.) Let X be a uniform $(0,1)$ random variable. Suppose that given X, the random variable Y is uniform on $(0,X)$. Given X and Y, the random variable Z is uniform on $(Y,1)$. What is the value of $E(Z)$?

(a.) $1/8$

(b.) $3/8$

(c.) $5/8$

(d.) $1/2$

(50.) Let $X_0 = 0$ and for $k \geq 1$ let X_k be a random variable with Binomial $\left(k, \frac{1}{2}\right)$ distribution. Let N be a Poisson random variable with mean 1. Assume that for every $k \geq 1$, X_k and N are independent and set $Y = X_N$. Given that $Y = 3$, what is the probability that $N = 3$?

(a.) $\frac{1}{6}e^{-1}$

(b.) $e^{-1/2}$

(c.) $\frac{1}{6}e^{-1/2}$

(d.) $\frac{1}{48}e^{-1/2}$

(51.) Let (N_t^1) and (N_t^2) be two independent Poisson processes with intensities a, b respectively, with $a > b$. Let $M_t = \max\{N_t^1, N_t^2\}$ and $M_t = \max\{N_t^1 - N_t^2, 0\}$. Then which of the following is true?

(a.) X is a Poisson process with intensity $a - b$

(b.) X is a birth-and-death process with birth-rate a and death-rate b

(c.) M is a Poisson process with intensity $\max(a, b)$

(d.) M is a pure birth process with birth rate $a + b$

(52.) Let $X_1, X_2, \ldots$ be i.i.d. Normal random variables with mean 2 and variance 3. Let N be a Poisson random variable with mean 4 that is independent of $\{X_1, X_2, \ldots\}$. Let $Y = X_1 + \ldots + X_N$ if $N \geq 1$ and $Y = 0$ if $N = 0$. What is the variance of Y?

(a.) 12

(b.) 16

(c.) 20

(d.) 28

(53.) Let $X_1, X_2, \ldots, X_n$ be i.i.d. $N(\theta, \sigma^2)$ random variables where σ^2 is known. Let $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$. The Minimum Variance Unbiased Estimator of $e^{2\theta}$ is given by (a.) $\exp\left(2\bar{X}-\frac{\sigma^2}{n}\right)$

(b.) $\exp\left(2\bar{X}-\frac{4\sigma^2}{n}\right)$

(c.) $\exp\left(2\bar{X}-\frac{2\sigma^2}{n}\right)$

(d.) $\exp\left(2n\bar{X}-\frac{2\sigma^2}{n}\right)$

(54.) Suppose that X follows a distribution with probability density function $f_\theta(x) \propto x^{\theta-1}(1-x)^{\theta-1}$; $0<x<1$, $\theta>0$. The uniformly most powerful critical region for testing $H_0:\theta=1$ against $H_1:\theta>1$ based on a single observation is of the form

(a.) $(1-\alpha, 1]$

(b.) $[0, \alpha)$

(c.) $\left(\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right)$

(d.) $\left[0, \frac{\alpha}{2}\right) \cup \left(1-\frac{\alpha}{2}, 1\right]$

(55.) Let $X_1, X_2, \ldots, X_n$ be i.i.d $N(\theta, 1)$ random variables. Suppose the prior distribution of θ is $N(0, \sigma^2)$. Suppose we have squared error loss function. Let $\bar{X}=\frac{\sum_{i=1}^{n} X_i}{n}$. Then, the Bayes estimator of θ is

(a.) $\frac{\bar{X}}{1+\sigma^2}$

(b.) $\bar{X}$

(c.) $\frac{n\bar{X}\sigma^2}{1+n\sigma^2}$

(d.) $\frac{n(\bar{X}+\sigma^2)}{1+n\sigma^2}$

(56.) Let X be a $p\times p$ matrix valued random variable having the Wishart distribution with parameters Σ (variance matrix) and m (degrees of freedom). Which of the following is necessarily true? ($A_{i,j}$ denotes the entry in i^{th} row and j^{th} column of a matrix A)

(a.) $cX_{1,2}$ has Chi-square distribution with $m-p+1$ degrees of freedom for a suitable constant c

(b.) $cX_{1,1}$ has Chi-square distribution with $m-p+1$ degrees of freedom for a suitable constant c

(c.) $Y = AX$ has the Wishart distribution with variance matrix $A\Sigma A^T$ and degrees of freedom m for any non-singular $p \times p$ matrix A

(d.) $k \times k$ sub-matrix of X also has a Wishart distribution for $k < p$

(57.) Consider a linear regression model of the form $y_i = \alpha + \beta x_i + \epsilon_i$ based on the data $\{(x_i, y_i) : i = 1, 2, \ldots, 50\}$ (here ϵ_i are the error terms). Assume that not all x_i are the same. Which of the following is an admissible value of the leverage of the 11th observation?

(a.) 0

(b.) 0.01

(c.) 0.1

(d.) 1.1

(58.) Suppose that $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are independent and they have a common distribution, which is uniform on the triangle with vertices $(0,0)'(\theta,0)$ and $(\theta,0)$, where $\theta > 0$. A sufficient statistic for θ is

(a.) $\max\limits_{1\le i\le n} X_i + \max\limits_{1\le i\le n} Y_i$

(b.) $\max\limits_{1\le i\le n} (X_i + Y_i)$

(c.) $\max\limits_{1\le i\le n} |X_i - Y_i|$

(d.) $\max\left\{\max\limits_{1\le i\le n} X_i, \max\limits_{1\le i\le n} Y_i\right\}$

(59.) A sample of size n is to be drawn from a population of N households to estimate the pro portion of the households which have more than one earning member. Let $0 < n < N$.

Sample I is drawn using Simple Random Sampling without replacement.

Sample II is drawn using Simple Random Sampling with replacement.

Let p_1 and p_2 denote the sample proportions in samples I and II respectively. Let $\sigma_i^2 = Var(p_i)$ for $i = 1, 2$.

Which of the following statements is true?

(a.) p_1 is an unbiased estimate of the population proportion but p_2 is not

(b.) p_2 is an unbiased estimate of the population proportion but p_1 is not

(c.) $\sigma_1^2 < \sigma_2^2$

(d.) $\sigma_2^2 < \sigma_1^2$

(60.) Consider the following linear Programming Problem.

Maximize $4x + 5y$

Subject to $2x + 3y \le 14$

$x + 2y \le 9$

$x + y \le 6$

$x \geq 0, y \geq 0$

What is the optimal value of the objective function?

(a.) 24

(b.) 26

(c.) 27

(d.) 25

PART-C
(Mathematical Sciences)

(61.) Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers.

Which of the following functions from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ are injective?

(a.) $f_1(m,n) = 2^m 3^n$

(b.) $f_2(m,n) = mn + m + n$

(c.) $f_3(m,n) = m^2 + n^3$

(d.) $f_4(m,n) = m^2 n^3$

(62.) Let $\{x_n\}$ be a sequence of positive real numbers. Which of the following statements are true?

(a.) If the two subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge, then the sequence $\{x_n\}$ converges

(b.) If $\{(-1)^n x_n\}$ converges, then the sequence $\{x_n\}$ converges

(c.) If $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists then $\{(x_n)^{1/n}\}$ is bounded

(d.) If the sequence $\{x_n\}$ is unbounded then every subsequence is unbounded

(63.) Let $f(x) = e^x$ for $x \in \mathbb{R}$. Which of the following statements are correct?

(a.) There is a real $C > 0$ such that $|f(x) - 1 - x| \leq Cx^2$ for all $x \in \mathbb{R}$

(b.) There is a real $C > 0$ such that $\left|f(x) - 1 - x - \frac{x^2}{2}\right| \leq C|x|^3$ for all $x \in [-1, 1]$

(c.) There is a real $C > 0$ such that $\left|f(x) - 1 - x - \frac{x^2}{2} - \frac{x^3}{3!}\right| \leq Cx^4$ for all $x \in \mathbb{R}$

(d.) There is a real $C > 0$ such that $\left|f(x) - 1 - x - \frac{x^2}{2} - \frac{x^3}{3!}\right| \leq Cx^4$ for all $x \in [-1, 1]$

(64.) Which of the following functions are uniformly continuous on $(0, 1)$?

(a.) $\frac{1}{x}$

(b.) $\sin\frac{1}{x}$

(c.) $x\sin\frac{1}{x}$

(d.) $\frac{\sin x}{x}$

(65.) Let x, y be real numbers such that $0 < y \le x$ and let n be a positive integer. Which of the following statements are true?

(a.) $ny^{n-1}(x-y) \le x^n - y^n$

(b.) $nx^{n-1}(x-y) \le x^n - y^n$

(c.) $ny^{n-1}(x-y) \ge x^n - y^n$

(d.) $nx^{n-1}(x-y) \ge x^n - y^n$

(66.) Consider the identity function $f(x) = x$ on $I := [0,1]$. Let P_n be the partition that divides I into n equal parts. If $U(f, P_n)$ and $L(f, P_n)$ are the upper and lower Riemann sums, respectively, and $A_n = U(f,P_n) - L(f,P_n)$ then

(a.) $\lim_{n\to\infty} nA_n = 0$

(b.) $\sum_{n=1}^{\infty} A_n$ is convergent

(c.) A_n is strictly monotonically decreasing

(d.) $\sum_{n=1}^{\infty} A_n A_{n+1} = 1$

(67.) For any two non-negative integers n, k define $f_{n,k}(x)$ on $[0,1]$ by

$$f_{n,k}(x) = \begin{cases} x^n \sin\left(\frac{\pi}{2x}\right) - x^k, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

In which of the following cases is the function $f_{n,k}$, a function of bounded variation?

(a.) For all $n \ge 1$ and for all $k \ge 0$

(b.) For all $n \ge 0$ and $k = 0$

(c.) For all $n \ge 1$ and for all $k \ge 2$

(d.) For all $n \ge 2$ and for all $k \ge 0$

(68.) Let $f(x,y) = (u(x,y), v(x,y)) : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function. Let A denote the matrix of the derivative of f at the origin $(0,0)$ with respect to the standard basis of $\mathbb{R}^2$. Assume $f(y,-x) = (v(x,y), -u(x,y))$ for all $(x,y) \in \mathbb{R}^2$. Which of the following statements are possibly true?

(a.) $A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

(b.) $A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

(c.) $A=\begin{pmatrix}1 & 2\\ -1 & -2\end{pmatrix}$

(d.) $A=\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}$

(69.) Which of the following functions f admit an inverse in an open neighbourhood of the point $f(p)$?

(a.) For $p=(1,0)$ and $f(x,y) = (x^3 \exp y + y - 2x, 2xy + 2x)$

(b.) For $p=(1,\pi)$ and $f(r,\theta)=(r\cos\theta, r\sin\theta)$

(c.) For $p=0$ and $f(x)=\begin{cases} x+2x^2\sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

(d.) For $p=0$ and $f(x)=\begin{cases} x^2\sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

(70.) Let $C_0(\mathbb{R})$ be the space of all continuous functions on $\mathbb{R}$ such that $\lim_{x\to\pm\infty} f(x)=0$. Let $C_0(\mathbb{R})$ be equipped with $\|\cdot\|$, the norm of uniform convergence. Let (f_n) be a sequence in $C_0(\mathbb{R})$ and $f\in C_0(\mathbb{R})$. Which of the following statements are correct?

(a.) If $f_n \to f$ uniformly on compact sets then $\|f_n - f\| \to 0$

(b.) If $\|f_n - f\| \to 0$ then $f_n \to f$ uniformly on compact sets

(c.) $f_n \to f$ uniformly on compact sets if and only if $\|f_n - f\| \to 0$

(d.) Neither of the two statements "$f_n \to f$ uniformly on compact sets" and "$\|f_n - f\| \to 0$" imply the other

(71.) Let A be an $m\times n$ matrix. Let $A(1,:),(:,1)$ and $A(1,1)$ be the matrices obtained from A by deleting row 1 deleting column 1 and deleting both row 1 and column 1 respectively. Which of the following hold?

(a.) $(\text{rank } A)-2 \le \text{rank } A(1,1) \le \text{rank } A$

(b.) $\text{rank } A(1,\ :)=\text{rank} A(:,1)$

(c.) $\text{rank } A = \text{rank } A(1,:) = \text{rank } A(:,1)$, then $\text{rank } A = \text{rank } A(1,1)$

(d.) $\text{rank } A(1,:)+\text{rank} A(:,1)+2 \ge 2\,\text{rank } A$

(72.) Let $M\in\mathbb{M}(\mathbb{R})$ with $M\neq 0, I_n$ but $M^2=M$. Which of the following statements are true?

(a.) Null (M) is the eigenspace of M corresponding to the eigenvalue 0

(b.) Let $X \in Col(M)$ with $X \neq 0$. Then X is an eigenvector of M corresponding to the eigenvalue 1

(c.) Let $X \notin \text{Null}(M)$. Then X is an eigenvector of M corresponding to the eigenvalue 1

(d.) $\mathbb{R}^n = \text{Col}(M) + \text{Null}(M)$

(73.) Let $W_1 = \left\{ \begin{bmatrix} a & b \\ -b & 2a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ and $W_2 = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ be two subspaces of $\mathbb{M}_2(\mathbb{R})$. Which of the following statements are true?

(a.) $W_1 \cap W_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

(b.) $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is a proper subset of $W_1 \cap W_2$

(c.) $W_1 + W_2 = \mathbb{M}_2(\mathbb{R})$

(d.) $W_1 + W_2$ is a proper subset of $\mathbb{M}_2(\mathbb{R})$

(74.) Let A be a 3×3 nilpotent matrix. Which of the following statements are necessarily true?

(a.) $(I + A)^n = I$ for some $n > 0$ where I is the identity matrix

(b.) The column space of A is $\{0\}$

(c.) The eigenvalues of A are roots of 1

(d.) A^3 is diagonalizable

(75.) A 100×100 matrix $A = (a_{i,j})$ is such that $a_{i,j} = i$ if $i + j = 101$ and $a_{i,j} = 0$ otherwise. Which of the following statements are true about A?

(a.) A is similar to diagonal matrix over $\mathbb{R}$

(b.) A is not similar to a diagonal matrix over $\mathbb{C}$

(c.) One of the eigenvalues of A is 10

(d.) None of the real eigenvalues of A exceeds 51

(76.) Let $\{v_1, v_2, v_3\}$ be an orthonormal basis of $\mathbb{R}^3$. Let V be the 3×3 matrix whose columns are v_1, v_2, v_3. Which of the following statements are necessarily true?

(a.) $VV^T = I$

(b.) $V^T V = I$

(c.) $V = V^T$

(d.) Determinant of V is not zero

(77.) Let $f : [0, 1] \to (0, 1)$ be a function. Which of the following statements are FALSE

(a.) If f is onto, then f is continuous

(b.) If f is continuous, then f is not onto (c.) If f is one-to-one, then f is continuous

(d.) If f is continuous, then f is not one-to-one

(78.) Let $\{y_1, y_2, y_3, y_4\}$ be an orthonormal basis of $\mathbb{R}^4$. Which of the following are orthonormal bases?

(a.) $\{y_1 + y_2, y_1 - y_2, y_3, y_4\}$

(b.) $\{y_3, y_4, y_1, y_2\}$

(c.) $\left\{\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}, y_3, y_4\right\}$

(d.) $\left\{\frac{3y_1 + 4y_2}{5}, \frac{4y_1 - 3y_2}{5}, y_3, y_4\right\}$

(79.) Consider the set

$S := \{\exp(2\pi i\theta) : \theta \text{ is a rational number}\}$.

For each $z \in S$ the set

$\{z^{n!} : n \text{ is a positive integer}\}$ is

(a.) Countable

(b.) Countably infinite

(c.) Uncountable

(d.) Finite

(80.) Fix a positive real number c. Consider the locus of all points $z \in C$ such that $\left|\frac{z-i}{z+i}\right| = c$. Which of the following statements are true?

(a.) If $c > 1$, the locus is a circle centered on the imaginary axis

(b.) If $c < 1$, the locus is a circle centered on the real axis

(c.) If $c = 1$, the locus is a straight line parallel to the imaginary axis

(d.) If $c = 1$, the locus is a straight line not passing through the origin

(81.) For $z \in \mathbb{C}$, let $\Re z$ denotes its real part. Let f be an entire function satisfying $|f(z)| \le |z||\Re z|$ on $\mathbb{C}$. Which of the following statements are true?

(a.) $f(0) = 0$

(b.) $f'(0) = 0$

(c.) The only entire function satisfying the given property is $f(z) \equiv 0$

(d.) There exists a non-constant entire function satisfying the given property

(82.) Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disc. Consider the family $\mathcal{F}$ of all holomorphic maps $f : \mathbb{D} \to \mathbb{D}$ such that $f(0) = 1/2$. For $f \in \mathcal{F}$, the possible values of $|f'(0)|$ are

(a.) $7/8$

(b.) 5/6
(c.) 3/4
(d.) 1

(83.) Let $\mathbb{N} = \{1,2,...\}$ denote the set of positive integers for $n \in \mathbb{N}$, let $A_n = \{(x,y,z) \in \mathbb{N}^3 : x^n + y^n = z^n \text{ and } z < n\}$.

Let $F(n)$ be the cardinality of the set A_n.

Which of the following statements are true?

(a.) $F(n)$ is always finite for $n \geq 3$
(b.) $F(2) = \infty$
(c.) $F(n) = 0$ for all n
(d.) $F(n)$ is non zero for some $n > 2$

(84.) For a positive integer n, let $\varphi(n)$ be the Euler's φ-function. Which of the following statements are true for $n > 3$?

(a.) $\varphi(n)$ can never divide n
(b.) $\varphi(n) | n \Rightarrow$ there exist integers x, y such that $nx + 6y = 1$
(c.) $\varphi(n) | n \Rightarrow$ n can have at most two distinct prime divisors
(d.) $\varphi(n) | \varphi(nk)$ for every positive integer k

(85.) For positive integers m and n, let $\gcd(m,n)$ denote their greatest common divisor. Let $m > n$ be such that $\gcd(m,n) = 1$. Which of the following statements are true?

(a.) $\gcd(m-n, m+n) = 1$
(b.) $\gcd(m-n, m+n)$ can have a prime divisor
(c.) There exists integers x, y such that $nx - my = 3$
(d.) $\gcd(m-n, m+n)$ can be an odd prime

(86.) Let G be the cyclic group of order 8 and $H = S_5$ be the permutation group of 5 elements. Which of the following statements are necessarily true?

(a.) There exists no nontrivial group homomorphism from G to H
(b.) There exists no injective group homomorphism from G to H
(c.) There exists no surjective group homomorphism from G to H
(d.) There are more than 20 different group homomorphisms from G to H

(87.) Which of the following statements are true for $\alpha \in \mathbb{R}$?

(a.) If α^3 is algebraic over $\mathbb{Q}$ then α is algebraic over $\mathbb{Q}$
(b.) α could be algebraic over $\mathbb{Q}[\sqrt{2}]$ but may not be algebraic over $\mathbb{Q}$

(c.) α need not be algebraic over any subfield of $\mathbb{R}$

(d.) There is an α which is not algebraic over $\mathbb{Q}[\sqrt{-1}]$

(88.) Let $p > 2019$ be a prime number. Consider the polynomial $f(x) = (x^2-3)(x^2-673)(x^2-2019)$. How many roots can f possibly have in the finite field $\mathbb{F}_p$?

(a.) 0
(b.) 2
(c.) 3
(d.) 6

(89.) Which of the following statements are true?

(a.) Any compact topological space is metrizable
(b.) Any continuous image of a Hausdorff topological space is Hausdorff
(c.) If $f: X \to Y$ is continuous and one-to-one, and Y is Hausdorff, then X is Hausdorff
(d.) Intersection of two connected subsets of $\mathbb{R}$ with the usual topology is either empty or connected

(90.) Let $X = \mathbb{N} \cup \{\infty, -\infty\}$. Let τ be the topology on X consisting of subsets U of X such that either $U \subset \mathbb{N}$ or $X \setminus U$ is finite. Let $A = \mathbb{N} \cup \{\infty\}$ and $B = \mathbb{N} \cup \{-\infty\}$. Which of the following subsets are compact?

(a.) A
(b.) $X \setminus A$
(c.) $A \cup B$
(d.) $A \cap B$

(91.) Let $f(x,y) = (y + x(1-x^2-y^2),\ -x+y\ (1-x^2-y^2))$ and consider the ODE $(\dot{x},\dot{y}) = f(x,y)$ with initial condition $(x(0), y(0)) = \left(0, \frac{1}{2}\right)$. Which of the following statements are true?

(a.) $x^2(t) + y^2(t) \to \infty$ as $t \to \infty$
(b.) $x^2(t) + y^2(t) \to 0$ as $t \to \infty$
(c.) as $x^2(t) + y^2(t) \to 0$ remains bounded as $t \to \infty$
(d.) $x^2(t) + y^2(t) \to 1$ as $t \to \infty$

(92.) Let x and y be continuously differentiable functions on $[0, \infty)$ that satisfy the respective initial value problems (ODEs)

$$\frac{dx}{dt} + (\sin t - 1)x = \log(1+t),\ x(0) = 1;$$

$$\frac{dy}{dt} + (\sin t - 1)y = t,\ y(2) = 2.$$

Define $z(t) = y(t) - x(t)$ for $t \geq 0$.

Which of the following statements are true?

(a.) $z(1) \le 1$

(b.) $z(2) > z(1)$

(c.) $z(1) > 1$

(d.) $z(2) \le z(1)$

(93.) Let $f:\mathbb{R}^2 \to \mathbb{R}$ be a locally Lipschitz function. Consider the system of ODEs given by $\dot{x}_1 = \sin\left(e^{x_2}\right)$, $\dot{x}_2 = f(x_1, x_2)$ with initial condition $(x_1(0), x_2(0)) = (1,1)$. Which of the following statements is true?

(a.) There is at most one local solution at time 0

(b.) There always exists a global solution defined in $[0,\infty)$

(c.) There might not be any solution around the time 0

(d.) There is at least one solution around time 0

(94.) Which of the following are solutions of the Laplace equation $u_{xx} + u_{yy} = 0$ in the unit disk $D = \{(x,y) : x^2 + y^2 < 1\}$?

(a.) $x^5 + 2x^2y^3 - y^5$

(b.) $x^2 + 2xy - y^2$

(c.) $\cos(y)e^x + \sin(x)e^y$

(d.) $\dfrac{1+x}{1+2x+x^2+y^2}$

(95.) Consider the partial differential equation (PDE) $x\left(\frac{\partial u}{\partial x}\right)^2 + y\left(\frac{\partial u}{\partial y}\right)^2 + (x+y)\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + 1 = 0$. Which of the following statements are true?

(a.) The general solution o f the PDE can be expressed in the form $u(x,y) = ax + by + \frac{1}{a+b}$, where a and b are arbitrary constants

(b.) The general solution of the PDE can be expressed in the form $u(x,y) = f(ax+by) + \frac{1}{a+b}$, where a and b are arbitrary constants and f is an arbitrary function

(c.) The Charpit's equations are

$$\frac{dx}{p^2+pq} = \frac{dy}{q^2+pq} = \frac{du}{p\left(p^2+pq\right)+q\left(p^2+pq\right)} = \frac{dp}{0} = \frac{dq}{0}$$

(d.) The Charpit's equations are

$$\frac{dx}{2px+(x+y)q-u} = \frac{dy}{2qy+(x+y)p-u}$$

$$=\frac{du}{p(2px+(x+y)q-u)+q(2qy+(x+y)p-u)}=\frac{dp}{0}=\frac{dq}{0}$$

(96.) Consider the numerical integration formula

$\int_{-1}^{1} g(x)\,dx \approx g(\alpha)+g(-\alpha)$, where $\alpha=(0.2)^{1/4}$.

Which of the following statements are true?

(a.) The integration formula is exact for polynomials of the form $a+bx$, for all $a,b\in\mathbb{R}$

(b.) The integration formula is exact for polynomials of the form $a+bx+cx^2$, for all $a,b,c\in\mathbb{R}$

(c.) The integration formula is exact for polynomials of the form $a+bx+cx^2+dx^3$, for all $a,b,c,d\in\mathbb{R}$

(d.) The integration formula is exact for polynomials of the form $a+bx+cx^3+dx^4$, for all $a,b,c,d\in\mathbb{R}$

(97.) Let S be the set of all continuous functions $f:[0,1]\to[0,\infty)$ that satisfy

$\int_0^1 x^2 f(x)\,dx=\frac{1}{2}\int_0^1 xf^2(x)\,dx+\frac{1}{8}$.

Which of the following statements are true?

(a.) S is an empty set

(b.) S has at most one element

(c.) S has at least one element

(d.) S has more than two elements

(98.) Consider the problem of extremising the functional $J(y)=\int_1^3 y(3x-y)\,dx$ with boundary conditions $y(1)=1$ and $y(3)=2$. Which of the following statements are true?

(a.) There is a unique extremal

(b.) $y(x)=\frac{3x}{2}$ is an extremal

(c.) $y(x)=\frac{x}{2}+\frac{1}{2}$ is an extremal

(d.) There are no extremals

(99.) Let B be the unit ball in $\mathbb{R}^3$ centered at origin. The Euler-Lagrange equation corresponding to the functional $I(u)=\frac{1}{2}\int_B |\nabla u|^2\,dx-\frac{1}{2}\int_B |u|^5\,dx$ is

(a.) $\Delta u=u^4$

(b.) $\Delta u=-u^4$

(c.) $\det(D^2u)=u^4$

(d.) $\Delta u=-|u|^3 u$

(100.) Let $K(x,y)$ be a kernel in $[0,1]\times[0,1]$, defined as $K(x,y)=\begin{cases} x(1-y), & 0\le x\le y\le 1 \\ y(1-x), & 0\le y\le x\le 1 \end{cases}$, and $\mathcal{K}$ be the associated integral operator. Then $\mathcal{K}: L^2([0,1])\to L^2([0,1])$

(a.) Is positive definite

(b.) Is self adjoint

(c.) Has a non-zero null space

(d.) Is onto

(101.) Let $K(x,y)$ be a kernel in $[0,1]\times[0,1]$, defined as $K(x,y)=\sin(xy)$. The integral equation $u(x)=\sin x+\int_0^1 K(x,y)u(y)\,dy$

(a.) Is uniquely solvable and the solution is differentiable

(b.) Has more than one differentiable solution

(c.) Has no solution

(d.) Is solvable and the solution is not differentiable

(102.) Consider a body of unit mass moving under an attractive central force. At a certain radius R, the body moves in a circular orbit. Which of the following are true?

(a.) The force must be equal to $-\frac{L^2}{R^3}$ where L is the angular momentum of the body

(b.) The force can be any strictly negative valued function of R

(c.) The force can only be- of the inverse square law form

(d.) The force cannot be of the form $-kR$

(103.) Let A and B be two events. Which of the following are necessarily correct?

(a.) $P(A\cup B)\le P(A)+P(B)$

(b.) $P(A\cup B)\ge P(A)+P(B)$

(c.) $P(A\cap B)=P(A)P(B)$

(d.) $P(A\cap B)\ge P(A)+P(B)-1$

(104.) Which of the following are true?

(a.) If X_1 and X_2 are independent uniform $(0,1)$ then X_1+X_2 is uniform $(0,2)$

(b.) If X is uniform $(0,n)$ then X/n is uniform $(0,1)$

(c.) If X_1 and X_2 are standard normal, then $(X_1+X_2)/\sqrt{2}$ is standard normal

(d.) If X is exponential with mean 1, then e^{-X} is uniform $(0,1)$

(105.) $(X_n, n\ge 0)$ is a Markov chain with state space $\{1,2,3,4,5\}$ and transition matrix $(p_{ij})_{1\le i,\, j\le 5}$ given by

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/3 & 0 & 0 & 0 & 2/3 \end{pmatrix}$$

Which of the following are true?

(a.) States 2 and 3 are absorbing

(b.) State 3 is recurrent

(c.) $\{1, 5\}$ form a recurrent class

(d.) State 4 is transient

(106.) Consider a Markov chain on a finite state space S. Suppose that the transition probability matrix P is such that the transpose of P is also a stochastic matrix. Then which of the following is necessarily true?

(a.) All states have the same period

(b.) The chain admits at most one stationary distribution

(c.) All states are recurrent

(d.) At least one state has period 1

(107.) Let X, Y be random variables with cumulative distribution functions F and G respectively. Assume that F is continuous. Which of the following functions of x are necessarily cumulative distribution functions (CDF)?

(a.) $\frac{1}{2}(1 - G(-x) + F(x))$

(b.) $\frac{1}{2}(1 - F(-x) + G(x))$

(c.) $F(x)G(x)$

(d.) $(1 - F(x))(1 - G(x))$

(108.) Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed Poisson (λ) random variables, where $\lambda > 0$ is unknown. Which of the following statements are true?

(a.) $\frac{1}{n}\sum_{i=1}^{n}(-1)^{X_i}$ is an unbiased estimator of $e^{-2\lambda}$

(b.) $\frac{1}{n}\sum_{i=1}^{n}(-1)^{X_i}$ is a consistent estimator for $e^{-2\lambda}$

(c.) $\left(\frac{n-2}{n}\right)^{\sum_{i=1}^{n}(-1)^{X_i}}$ is the minimum variance unbiased estimator of $e^{-2\lambda}$

(d.) $\left(\frac{n-2}{n}\right)^{\sum_{i=1}^{n}(-1)^{X_i}}$ is a consistent estimator of $e^{-2\lambda}$

(109.) Let X_1, X_2 be i.i.d. random variables which are uniformly distributed or the interval $(0,\theta)$. Consider the problem of testing $H_0:\theta=1$ versus $H_1:\theta=2$. Which of the following tests have size ≤ 0.5 and power ≥ 0.5?

(a.) Reject H_0 if and only if $\max\{X_1, X_2\}\ge 1.6$

(b.) Reject H_0 if and only if $\min\{X_1, X_2\}\ge 1$

(c.) Reject H_0 if and only if $X_1 \ge 0.4$

(d.) Reject H_0 if and only if $X_1 - X_2 \ge 0$

(110.) Let $X_1, X_2, \ldots, X_n$ be i.i.d. $N(0,\sigma^2)$ random variables. Which of the following are sufficient for σ^2 ?

(a.) $T_1(X_1, X_2, \ldots, X_n) = (X_1, X_2, \ldots, X_n)$

(b.) $T_2(X_1, X_2, \ldots, X_n) = (X_1^2, X_2^2, \ldots, X_n^2)$

(c.) $T_3(X_1, X_2, \ldots, X_n) = (X_1^2 + X_2^2 + \ldots + X_n^2)$

(d.) $T_3(X_1, X_2, \ldots, X_n) = (X_1^2 + X_2^2, X_3^2 + X_4^2 + \ldots + X_n^2)$

(111.) Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be two independent random samples from a common continuous distribution F. Let W be the sum of ranks of X observations in the combined ranking of all the $N = n+m$ observation. Which of the following are true?

(a.) $E(W) = \dfrac{n(N+1)}{2}$

(b.) $E(W) = \dfrac{N(N+1)}{2}$

(c.) The distribution of W is symmetric about $\dfrac{n(N+1)}{2}$

(d.) The distribution of W does not depend on F

(112.) A sample of size 10 is selected at random from a lot containing 20 items of a manufactured product. Suppose the total number of defectives D in the lot is unknown. Let X be the number of defectives in the sample. We wish to test $H_0: D=7$ against the alternative $H_1: D>7$. Consider the test: Reject H_0 if $X \ge k$ where k is determined such that

$P_{H_0}(X \ge k) \le 0.05 < P_{H_0}(X \ge k-1)$.

Which of the following are true?

(a.) The test is Uniformly Most Powerful test of its size

(b.) The power function of the test is increasing in D

(c.) The power function of the test is decreasing in D

(d.) The test is unbiased

(113.) Consider the Gauss-Markov Model $\mathbf{Y}_{4\times1} = \mathbf{X}_{4\times4}\boldsymbol{\beta}_{4\times1} + \varepsilon_{4\times1}$, where $\boldsymbol{\beta} = (\beta_1\beta_2\beta_3\beta_4)$ and $\varepsilon \sim N(\mathbf{0}, \sigma^2\mathbf{I})$. Two choices for $\mathbf{X}$ are given below:

Case-1: $\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Case-2: $\mathbf{X} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

Which of the following statements are true?

(a.) The contrast $\beta_1 - \beta_2$ is estimable in Case I, but not in Case 2

(b.) The contrast $\beta_1 - \beta_2$ is estimable in both cases and $Var\left(\hat{\beta}_1 - \hat{\beta}_2\right)$ is larger in Case 1 then in Case 2

(c.) The contrast $\beta_1 - \beta_2$ is estimable in Case 1, but not in Case 2

(d.) The contrast $\beta_1 - \beta_2$ is estimable in both cases and $Var\left(\hat{\beta}_1 - \hat{\beta}_3\right)$ is larger in Case 1 then in Case 2.

(114.) Consider a population which is distributed according to $f_\theta(x)$. Here f_θ depends on an unknown parameter θ and denotes either a probability mass function or a probability density function.

Consider a random sample $X_1, X_2, \ldots, X_n$ from this population. Let $\bar{X}$ denote the corresponding sample mean. Among the following identify those cases for which $\bar{X}$ is the Minimum Variance Unbiased Estimator of θ.

(a.) $f_\theta(x) \propto \exp\{-\theta x\}$ for $x \geq 0$; $\theta > 0$

(b.) $f_\theta(x) \propto \exp\{-(\theta - x)^2\}$ for $-\infty < x < \infty$; $-\infty < \theta < \infty$

(c.) $f_\theta(x) \propto \theta^x / x!$ for $x = 0, 1, 2, \ldots$; $\theta > 0$

(d.) $f_\theta(x) \propto \theta^x$ for $x = 0, 1, 2, \ldots$; $0 < \theta < 1$

(115.) Let $(X_1, Y_1), (X_2, Y_2) \ldots, (X_n, Y_n), n \geq 4$ be a random sample from the bivariate normal distribution with $E(X_1) = \mu_1$, $E(Y_1) = \mu_2$, $Var(X_1) = Var(Y_1) = \sigma^2$ and the correlation coefficient between X_1 and Y_1 equal to ρ. All four parameters μ_1, μ_2, σ^2 and ρ are unknown. Also, let $S_{xx} = \sum_{i=1}^{n}\left(X_i - \bar{X}\right)$, $S_{yy} = \sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2$, $S_{xy} = \sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)$.

Which of the following are maximum likelihood estimators of ρ?

(a.) $\dfrac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$

(b.) $\dfrac{\sum_{i=1}^{n} X_i Y_i}{\sqrt{\left(\sum_{i=1}^{n} X_i^2\right)\left(\sum_{i=1}^{n} Y_i^2\right)}}$

(c.) $\dfrac{2S_{xy}}{S_{xx}+S_{yy}}$

(d.) $\dfrac{2S_{xy}}{S_{xx}+S_{yy}-2S_{xy}}$

(116.) Let $X_1, X_2, \ldots, X_n$ be a random sample from normal distribution with unknown mean μ and variance 1. Let $\bar{X}$ be the sample mean and $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$.

Which of the following are true?

(a.) $\bar{X}$ is complete sufficient for μ

(b.) $\bar{X}$ is minimal sufficient for μ

(c.) $\bar{X} - X_{(1)}$ is an ancillary statistic

(d.) $Cov\left(\bar{X} - X_{(1)}\right) = 0$

(117.) In a BIBD (Balanced Incomplete Block Design) with υ treatments, b blocks, let t_j denote the effect of j^{th} treatment, b_t denote the effect of t^{th} block $1 \le j \le \upsilon$, $1 \le t \le b$.

Which of the following statements are necessarily true?

(a.) t_j is estimable for all $1 \le j \le \upsilon$

(b.) $t_j - b_t$ is estimable for all $1 \le j \le \upsilon$, $1 \le t \le b$

(c.) $t_j - t_i$ is estimable for all $1 \le i < j \le \upsilon$

(d.) $t_j + t_i$ is estimable for all $1 \le i < j \le \upsilon$

(118.) Suppose X denotes the lifetime of a system and follows the distribution with cumulative distribution function $F(t) = 1 - e^{-t\gamma}$, $t > 0$, $\gamma > 0$. Let $r(t)$ be the hazard (failure) rate of X. Which of the following are true?

(a.) $r(t)$ is increasing in t if $\gamma > 1$

(b.) $r(t)$ is decreasing in t if $\gamma > 1$

(c.) $r(t)$ is constant in t if $\gamma = 1$

(d.) $r(t)$ is decreasing in t if $0 < \gamma < 1$

(119.) Let $(X_n, n \ge 1)$ be i.i.d. random variables with mean zero and variance one. Which of the following are true as $n \to \infty$?

(a.) $\dfrac{X_1 + X_2 + \ldots + X_n}{\sqrt{n+1}}$ converges in distribution to standard normal (b.) $\frac{X_1 - X_2 + \ldots + (-1)^{n+1} X_n}{\sqrt{n}}$ converges in distribution to standard normal

(c.) $\frac{X_1 + X_2 + \ldots + X_n}{n+1}$ converges to zero in probability

(d.) $\frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}$ converges to zero in probability

(120.) Consider a M/M/1 queue at the steady state with traffic intensity $\rho \in (0,1)$. Let $f(\rho)$ be the expected waiting time of a customer in the queue. Then

(a.) f is increasing in ρ

(b.) f is decreasing in ρ

(c.) f attains its maximum at $\rho = \frac{1}{2}$

(d.) f is a smooth function in $(0,1)$